

Classical Solution for the Bounce Up to Second Order

Hatem Widyan *and Mashhoor Al-Wardat

Department of Physics

Al-Hussein Bin Talal University

P.O.Box 20, 71111, Ma'an, Jordan

Abstract: Scalar field theory with asymmetric potential is studied for ϕ^4 theory with ϕ^3 symmetry breaking. The equations of motion are solved analytically up to the second order to get the bounce-solution.

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1 Introduction

Vacuum decay is an old subject in field theory [1]. Coleman and Callan [2] showed that a quantum tunneling process from a false vacuum to a true vacuum can be realized via the nucleation of a true vacuum bubble in the surrounding of a false one. Coleman and De Luccia [3] found that gravity has a significant effect on the vacuum decay process.

In semiclassical approximation, the decay rate per unit volume is given by an expression of the form

$$\Gamma = A e^{-S_E/\hbar} [1 + O(\hbar)], \quad (1)$$

where S_E is the Euclidean action for the bounce: the classical solution of the equation of motion with appropriate boundary conditions and the prefactor A , comes from Gaussian functional integration over small fluctuations around the bounce, and has been discussed in

*E-mail : widyan@ahu.edu.jo

Ref. [3, 4, 5]. The solution of the equation of motion looks like a bubble in four dimensional Euclidean space with radius R and, thickness proportional to the coefficient of the symmetry breaking term in the potential. If there is more than one solution satisfying the boundary conditions, then that with the lowest S_E dominates Eq. (1).

Recently, some authors have discussed the vacuum decay in different situations such as: different scalar field theories [6], nonminimal coupling between the scalar field and curvature scalar [7], DBI action [8], and non-thin-wall limit [9], etc. The finite temperature effect on the false vacuum decay process has also been discussed by Linde et. al. [10], where one should look for the $O(3)$ -symmetric solution due to periodicity in the time direction with the inverse temperature period T^{-1} , instead of the $O(4)$ -symmetric solution at zero temperature. The cosmological applications of false vacuum decay process have been applied to various inflation cosmological models [11].

In general, it is not possible to find an analytical solution of the field equations for a finite difference of the potential between two minima. An approximation, where the field equations can be solved exactly, is the "thin wall approximation" (TWA) [2]. An analytical calculation of the nucleation rate for first order phase transitions beyond the TWA of the standard Ginzburg-Landau potential with ϕ asymmetric term were studied by Münster and Rotsch [5]. In this paper, we extend our earlier work for the ϕ^4 theory with ϕ^3 symmetry breaking term [6], and we calculate the bounce and the radius of the bubble up to second order by expanding the bubble solution in powers of the asymmetry parameter.

The paper is organized as follows. In Sec. 2 we present the Euclidean action and the equation of motion of the scalar field ϕ . In Sec. 3 and 4 we calculate the bounce and radius of the bubble as well as the action. Sec. 5 includes our conclusion and discussion.

2 Equation of the motion and the action

Let us consider a three-dimensional scalar field theory with a Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial_\mu \phi - U_s(\phi), \quad (2)$$

where $U_s(\phi)$ is a symmetric double-well potential having two degenerate minima at $\phi = \pm 1$ and has the form

$$U_s(\phi) = \frac{1}{2}(\phi^2 - 1)^2. \quad (3)$$

To shift mutually the minima, we introduce a small asymmetry term proportional to δ :

$$U(\phi) = \frac{1}{2}(\phi^2 - 1)^2 + \delta\phi^3, \quad (4)$$

where $0 \leq \delta \leq 2$. The parameter δ fixes the asymmetry of the potential. In particular, the difference between the values of the potential ϕ_{\pm} is

$$U(\phi_+) - U(\phi_-) = 2\delta + O(\delta^3) \quad (5)$$

The shift of the minima in terms of the asymmetry parameter δ is given by

$$\phi_{\pm} = \pm 1 - \frac{3}{4}\delta \pm \frac{9}{32}\delta^2 + O(\delta^4) \quad (6)$$

The field equation for a radially symmetric field is

$$-\frac{d^2\phi}{dr^2} - \frac{2}{r}\frac{d\phi}{dr} + 2\phi^3 - 2\phi^2 + 3\delta\phi^2 = 0, \quad (7)$$

with the boundary conditions

$$\phi \rightarrow \phi_+ \quad \text{as} \quad r \rightarrow \infty, \quad \frac{d\phi}{dr} = 0 \quad \text{at} \quad r = 0. \quad (8)$$

Using the expression of ϕ we can calculate the Euclidean action of the bounce, which is given by

$$\begin{aligned} S_E(\phi) &= \int d^3x \left[\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + U(\phi) \right] \\ &= 4\pi \int_0^\infty dr r^2 \left(\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + \left[\frac{1}{2}(\phi^2 - 1)^2 - \frac{1}{2}(\phi_+ - 1)^2 \right] + \delta(\phi^3 - \phi_+^3) \right). \end{aligned} \quad (9)$$

3 The bounce solution

Before applying the systematic approach to solve Eq. (7), let us consider the thin wall approximation. This usually happens when δ is small, and gives a solution nearly equals to

-1 inside a sphere of radius R and nearly equal to $+1$ outside it. The region where ϕ differs significantly from these values is a kink of the form

$$\phi(r) = \tanh(r - R), \quad (10)$$

which is the solution of the field equation

$$-\frac{d^2\phi}{dr^2} - \frac{2}{r} \frac{d\phi}{dr} + 2\phi^3 - 2\phi^2 = 0. \quad (11)$$

One can easily show that the energy of the bubble (the action of the bounce solution) of radius R in the thin wall approximation (see for example [2]) is given by

$$S_E(R) = -\frac{4}{3}\pi R^3 \epsilon + 4\pi R^2 S_1, \quad (12)$$

where $\epsilon = 2f$ and S_1 is the bubble-wall surface energy (surface tension), and is given by

$$S_1 = \int dr \left\{ \frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + U_s(\phi) \right\}. \quad (13)$$

The critical bubble radius R , for which the energy is stationary, is written in terms of S_1 and ϵ as

$$R = \frac{2S_1}{\epsilon} = \frac{4}{3} \frac{1}{\delta}, \quad (14)$$

whence it follows that

$$S_E = \frac{16\pi S_1^3}{3\epsilon^2} = \frac{256\pi}{81} \frac{1}{\delta^2}. \quad (15)$$

Note that for in the thin wall approximation (i.e. small δ), we have exact analytical result for the radius as well as for the action. The radius is large and therefore the wall is indeed thin compared to the size of the bubble.

For finite δ , the solution of the equation of motion Eq. (7) can't be written in a closed form. Following the approach in [5], the solution is constructed by means of an expansion in powers of δ .

Since the bounce-solution is centered around the radius R , then it is more convenient to write the equation of motion in terms $\xi = r - R$. Hence Eq. (7) becomes

$$-\frac{d^2\phi}{d\xi^2} - \frac{2}{\xi + R} \frac{d\phi}{d\xi} + 2\phi^3 - 2\phi + 3\delta\phi^2 = 0 \quad (16)$$

Depending on the thin wall approximation, we may write a Laurent series as an ansatz for the critical radius,

$$R = \frac{a_{-1}}{\delta} + a_0 + a_1\delta + a_2\delta^2 + \dots = \sum_{i=-1}^{\infty} a_i\delta^i. \quad (17)$$

The factor of the first derivative in the equation of motion becomes

$$\frac{1}{\xi + R} = \frac{1}{\xi + \sum_{i=-1}^{\infty} a_i\delta^i} = \frac{\delta}{a_{-1}} - \frac{\xi + a_0}{a_{-1}^2}\delta^2 + \frac{(\xi + a_0)^2 - a_1a_{-1}}{a_{-1}^3}\delta^3 + O(\delta^4). \quad (18)$$

Hence the expansion of the field equation in powers of δ reads

$$-\frac{d^2\phi}{d\xi^2} - \frac{2}{a_{-1}}\delta\frac{d\phi}{d\xi} + \frac{2(\xi + a_0)}{a_{-1}^2}\delta^2\frac{d\phi}{d\xi} + 2\phi^3 - 2\phi + 3\delta\phi^2 + O(\delta^3) = 0. \quad (19)$$

Now, the solution of the above equation is obtained perturbatively up to the second order by means of the expansion

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + O(\delta^3). \quad (20)$$

For the zero order of δ the equation of motion is

$$-\frac{d^2\phi_0}{d\xi^2} + 2\phi_0^3 - 2\phi_0 = 0, \quad (21)$$

which has kink's solution

$$\phi_0(\xi) = \tanh\xi. \quad (22)$$

Note that the solution satisfies the boundary conditions given in Eq. (8), i.e., $\phi_0(\infty) = +1$ and $\phi'_0(0) = 0$.

The equation of motion for the first order of δ

$$-\frac{d^2\phi_1}{d\xi^2} - (6\operatorname{sech}^2\xi - 4)\phi_1 = \left(\frac{2}{a_{-1}} + 3\right)\operatorname{sech}^2\xi - 3. \quad (23)$$

The homogenous part of Eq. (23) has the following two independent solutions

$$\begin{aligned} \phi_{1,H_1}(\xi) &= -C_1 \operatorname{sech}^2\xi \\ \phi_{1,H_2}(\xi) &= C_2 \left(\frac{3}{2}\xi \operatorname{sech}^2\xi + \frac{3}{2}\tanh\xi + \sinh\xi \cosh\xi \right). \end{aligned}$$

The particular solution of the equation which satisfies the boundary conditions for $\xi \rightarrow \infty$ is

$$\phi_{1,S} = C_3 + C_4 \operatorname{sech}^2\xi$$

Substituting $\phi_{1,S}$ in Eq. (23), we get

$$a_{-1} = \frac{4}{3}, \quad (24)$$

which fixes the leading coefficient in R , Eq. (17), and

$$\phi_{1,S} = -\frac{3}{4}. \quad (25)$$

The general bounce-solution is given in the first order of δ by

$$\phi_1(\xi) = -C_1 \operatorname{sech}^2 \xi + C_2 \left(\frac{3}{2} \xi \operatorname{sech}^2 \xi + \frac{3}{2} \tanh \xi + \sinh \xi \cosh \xi \right) - \frac{3}{4}. \quad (26)$$

Note that the constant term reflects the shift of the minimum given in Eq. (6). The term proportional to C_2 diverges for large ξ , hence C_2 must be zero. The first term corresponds to the derivative of the zero-order of the bounce-solution, and is related its translation degree of freedom. To see this, we expand the zero-order solution around $\delta = 0$ in a Taylor series as

$$\tanh(\xi - C_1 \delta) = \tanh \xi - C_1 \delta \operatorname{sech}^2 \xi - C_1^2 \delta^2 \operatorname{sech}^3 \xi \sinh \xi + O(\delta^3). \quad (27)$$

Note that C_1 corresponds to the parameter a_1 in the ansatz for the radius Eq. (17). The homogenous solution proportional to C_1 is already taken into consideration by this ansatz, and should not consider in the the following calculations. Hence, we remain with

$$\phi_1(\xi) = -\frac{3}{4}. \quad (28)$$

Note that ϕ_1 satisfies the boundary conditions given in Eq. (8).

The equation of motion for the second order of δ

$$-\frac{d^2 \phi_2}{d\xi^2} + (6 \operatorname{sech}^2 \xi - 2) \phi_2 = -\frac{9}{8} \xi \operatorname{sech}^2 \xi - \frac{3}{2} a_0 \operatorname{sech}^2 \xi + \frac{9}{8} \tanh \xi, \quad (29)$$

which has a general solution

$$\begin{aligned} \phi_2(\xi) &= -D_1 \operatorname{sech}^2 \xi + D_2 \left(\frac{3}{2} \xi \operatorname{sech}^2 \xi + \frac{3}{2} \tanh \xi + \sinh \xi \cosh \xi \right) + \xi \left(\frac{3}{16} + \frac{3}{16} \cosh^2 \xi - \frac{15}{64} \operatorname{sech}^2 \xi \right) \\ &- \operatorname{Log} \cosh \xi \left(\frac{9}{32} \xi \operatorname{sech}^2 \xi + \frac{9}{32} \tanh \xi + \frac{3}{16} \sinh \xi \cosh \xi \right) + \frac{3}{64} \tanh \xi - \frac{3}{16} \sinh \xi \cosh \xi \\ &+ a_0 \left(\frac{9}{2} + 4 \sinh^2 \xi \tanh^2 \xi \right) + \frac{9}{32} \operatorname{sech}^2 \xi T(\xi), \end{aligned} \quad (30)$$

where we define

$$T(\xi) = \int_0^\xi \xi' \tanh \xi' d\xi'. \quad (31)$$

The general solution has a converging terms as well as diverging terms for large values of ξ . As the same argument for ϕ_1 , $D_1 = 0$. The term proportional to $a_0 = 0$ is symmetric in ξ , while the diverging homogenous solution of the bounce is antisymmetric, so we set $a_0 = 0$ and we assume $D_2 = \frac{3}{16} - \frac{3}{16} \text{Log}2$ in order to get rid of the diverging terms. Therefore the converging solution is given by

$$\begin{aligned} \phi_2(\xi) = & -\frac{9}{32} \xi (\tanh \xi - 1) + \frac{3}{32} \xi (\cosh \xi - \sinh \xi)^2 - \frac{9}{32} \xi^2 \text{sech}^2 \xi + \frac{3}{64} \xi \text{sech}^2 \xi + \frac{21}{64} \tanh \xi \\ & - \text{Log}(1 + e^{-2\xi}) \left(\frac{9}{32} \xi \text{sech}^2 \xi + \frac{9}{32} \tanh \xi + \frac{3}{16} \sinh \xi \cosh \xi \right) + \frac{9}{32} \text{sech}^2 \xi T(\xi). \end{aligned} \quad (32)$$

In order check whether ϕ_2 is an acceptable solution or not, we find its values at a large values of ξ , and its derivative at $\xi = -R$. It can be easily shown that

$$\phi_2(\xi \rightarrow \infty) = \frac{9}{32}, \quad (33)$$

which reflects the shift of the minimum given in Eq. (6), and

$$\phi'_2(\xi = -R) = 0. \quad (34)$$

Hence ϕ_2 satisfies the boundary conditions given in Eq. (8), which means that it is an acceptable solution.

Whereas the first order solution corresponds to the shift of the minimum and the critical radius, the second order solution describes a true deformations of the bubble. The boundary condition at $r = 0$, i.e. $\xi = -R$, is fulfilled order by order in δ . For example, the leading order solution yields

$$\phi'_0(-R) = e^{-2/\delta} (4 + O(\delta)), \quad (35)$$

which vanishes to all orders in δ . Similar observations hold in higher orders.

The bounce solution up to the second order of δ is

$$\begin{aligned} \phi(\xi) = & \phi_0 + \delta \phi_1 + \delta^2 \phi_2 + O(\delta^3). \\ = & \tanh \xi - \delta \frac{3}{4} + \delta^2 \left(-\frac{9}{32} \xi (\tanh \xi - 1) + \frac{3}{32} \xi (\cosh \xi - \sinh \xi)^2 - \frac{9}{32} \xi^2 \text{sech}^2 \xi + \frac{3}{64} \xi \text{sech}^2 \xi \right. \\ & + \frac{21}{64} \tanh \xi - \text{Log}(1 + e^{-2\xi}) \left(\frac{9}{32} \xi \text{sech}^2 \xi + \frac{9}{32} \tanh \xi + \frac{3}{16} \sinh \xi \cosh \xi \right) \\ & \left. + \frac{9}{32} \text{sech}^2 \xi T(\xi) \right) + O(\delta^3). \end{aligned} \quad (36)$$

4 The action and the critical radius

With the expression for ϕ we can calculate the action of the bounce, which is given by Eq. (9) as

$$\begin{aligned} S_E(\phi) &= 4\pi \int_0^\infty dr r^2 \left(\frac{1}{2} \phi'^2 + \left[\frac{1}{2} (\phi^2 - 1)^2 - \frac{1}{2} (\phi_+ - 1)^2 \right] + \delta(\phi^3 - \phi_+^3) \right) \\ &= S_0 + \delta S_1 + \delta^2 S_2 + O(\delta^3). \end{aligned} \quad (37)$$

The integrands are centered around the critical radius $r = R$. The integration range in ξ can be extended to the whole real axis [5]. Hence

$$\begin{aligned} S_0 &= 4\pi \int_{-\infty}^\infty d\xi (\xi + R)^2 \left(\frac{1}{2} \phi_0'^2 + \left[\frac{1}{2} (\phi_0^2 - 1)^2 - \frac{1}{2} (\phi_{0+} - 1)^2 \right] \right) \\ &= 2\pi \left(\frac{8}{3} R^2 - \frac{12 - 2\pi^2}{9} \right). \end{aligned} \quad (38)$$

$$\begin{aligned} S_1 &= 4\pi \int_{-\infty}^\infty d\xi (\xi + R)^2 \left(\phi_0' \phi_1' + 2\phi_0 \phi_1 (\phi_0^2 - 1) + (\phi_0^3 - 1) \right) \\ &= 2\pi \left(-\frac{4}{3} R^3 + \left(2 - \frac{\pi^2}{3} \right) R \right). \end{aligned} \quad (39)$$

$$\begin{aligned} S_2 &= 2\pi \int_{-\infty}^\infty d\xi (\xi + R)^2 \left(\phi_1'^2 + 2\phi_0' \phi_2' + 6\phi_0^2 \phi_1^2 - 2\phi_1^2 + 4\phi_0 \phi_2 (\phi_0^2 - 1) - \frac{9}{4} + \frac{9}{2} (1 - \phi_0^2) \right) \\ &= 2\pi \left(-\frac{9}{4} R^2 + 1.01528 \right). \end{aligned} \quad (40)$$

So, all together, the action is given up to the second order of δ as

$$S_E = 2\pi \left[\frac{8}{3} R^2 - \frac{12 - 2\pi^2}{9} + \delta \left(-\frac{4}{3} R^3 + \left(2 - \frac{\pi^2}{3} \right) R \right) + \delta^2 \left(-\frac{9}{4} R^2 + 1.01528 \right) \right]. \quad (41)$$

To find an expression for the critical radius R , we minimize the action with respect to R , i.e.,

$$\frac{dS_E}{dR} = 0. \quad (42)$$

Explicit calculation of the coefficients leads to two more terms in the Laurent series in R :

$$R = \frac{4}{3} \frac{1}{\delta} + 0 + \delta \left(\frac{11}{64} - \frac{\pi^2}{16} \right) + 0. \delta^2 + O(\delta^3), \quad (43)$$

so that the bubble is now completely determined to the second order. Hence, the action follows as

$$S_E = 2\pi \left[\frac{128}{81} \frac{1}{\delta^2} - \left(\frac{8}{3} + \frac{2\pi^2}{9} \right) + O(\delta^2) \right]. \quad (44)$$

For the thin wall approximation, we have obtained an explicit values of the radius and the bounce which are given in Eq. (14) and Eq. (15) respectively, these are exactly the values of the leading terms in Eq. (43) and Eq. (44).

To check the validity of our analytical results, we have compared our analytical results with the thin-wall approximation for different values of δ . Figure 1 shows a plot of S_E/S_{TWA} , we can see that as far as $\delta < 0.1$, the ratio is of order 1, and for $\delta > 0.1$, the ratio is less than 1. The result is physically expected, since the maximum value of the action is in the case of thin-wall approximation. Similar observation also for the radius as shown in figure 2. We have also solved the equation of motion (Eq. (7)) numerically. Figure 3 shows the ratio Euclidean action of the numerical results to the analytical results. We notice from the figure that as far as $\delta < 0.1$, there is a good agreement between them, and for $\delta > 0.1$, our analytical results starts deviates from the numerical ones.

5 Conclusion

By expanding all of the quantities in power of δ , we found the bounce solution, the critical radius and the action analytically beyond the thin wall approximation. We also showed that the leading terms correspond to the thin wall approximation.

The comparison between our analytical results and the numerical ones showed a good correlation up to $\delta = 0.1$, and after that the analytical results starts deviates form the numerical ones. Also our analytical results are consistent with the TWA results up to $\delta = 0.1$.

To calculate the nucleation rate, we have to find the fluctuation determinant in powers of δ , this will be determined in future work.

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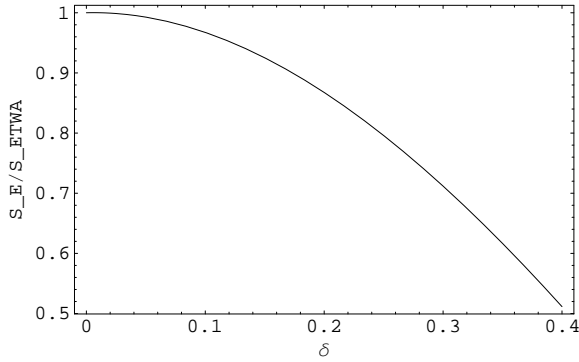


Figure 1: Ratio of the analytical action to the action in the thin-wall approximation: S_E/S_{ETWA} vs δ

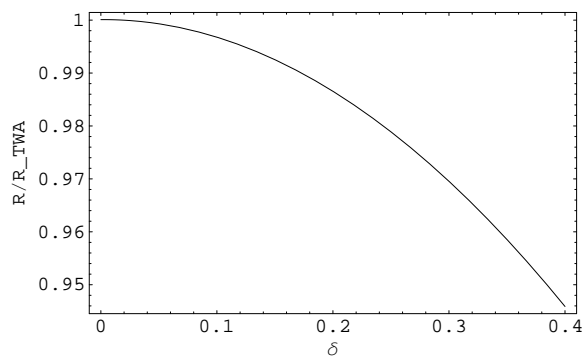


Figure 2: Ratio of the analytical radius of the bubble to the radius in the thin-wall approximation: R/R_{TWA} vs δ

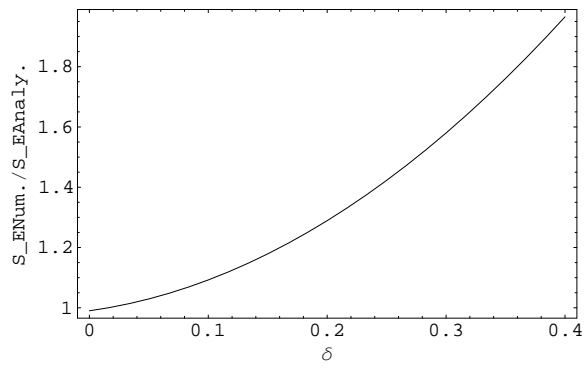


Figure 3: Ratio of the numerical action to the analytical action: $S_{ENum.}/S_{EAnaly.}$ vs δ